CHAPTER XI

MATHEMATICAL KNOWLEDGE

WE have now completed a definite stage in our journey. We began with the given of the solitary mind. We have advanced to a point at which all the details of the everyday world of common knowledge stand out before us. I now know that other minds exist as well as my own. I know that what I see, hear, and touch is not merely my private and evanescent percept, but consists of 'things' which are public in the sense that other people as well as myself are aware of them; that these things exist when no one is aware of them; and that they have their being in a single continuous three-dimensional public space and in a single continuous public time. In reaching these results we have analysed in principle the logical evolution of the whole of what I call 'common knowledge', by which I mean that knowledge of the world which has been attained by the whole human race, by all normal human beings without any special training or education, by uneducated people, and even by savages.

We have thus finished our examination of 'common knowledge'. We must now pass on to the epistemological analysis of the more advanced kinds of knowledge, of the knowledge which has been evolved by specially civilized and cultured peoples. This will include, for example, science, mathematics, and philosophy. And I will call it in general 'advanced knowledge' to distinguish it from 'common knowledge'.

I need hardly say that between these two kinds of knowledge there is no hard and fast line. Much less is there any difference in essential epistemological character. Indeed one of the main lessons which I am attempting to drive home is that all knowledge is of a piece, and that the epistemological features of the highest truths of science are one with those of the humblest perceptual knowledge. The distinction between common and advanced knowledge is not one which is intended to have the slightest scientific basis. It is merely made for the purpose of having before our minds some rough idea of how far we have travelled and how far we have still to go. Moreover, as I am about to explain, the point we have reached is one at which some change of method is necessary.

Our method so far has been to proceed logically step by step from the elementary certitudes of the given right up to the establishment of all the main features of the common world. We hoped to establish a connected and rigorous chain of logical propositions stretching from the first point of the development to the last. We hoped to leave out nothing essential, but to examine in detail the whole field of 'common knowledge'.

It is obvious that we cannot hope to carry out this comprehensive procedure when it comes to the consideration of 'advanced knowledge'. To do so would require, for example, a complete and encyclopaedic elaboration of all the knowledge contained in all the sciences, of all mathematics, of all history, &c. Theoretically speaking, we hold that the entire fabric of human knowledge might be logically developed in this way from its most primitive beginnings. Its development has been continuous. We should be able to advance logically step by step from the elementary certitudes of the given up to the highest pinnacles of the most advanced science. We should be able to show how and why every brick in the vast structure of knowledge has been laid where it is by the creative mind. But to do this would obviously be an impossible task in practice. It would necessitate not only an examination of the whole field of human knowledge, but its articulation in the form of a logical evolution.

All we can actually do is to examine very briefly a few selected but typical pieces of scientific, logical, mathematical, and other knowledge. And the objectives which we shall keep before us will be the following: (1) We shall attempt to show that advanced knowledge possesses in the main the same epistemological characters as we have already found to be possessed by common knowledge, especially the feature of advancing by means of mental constructions. This will bear out our view that knowledge is all of a piece and governed by the same principles throughout. (2) We shall endeavour to ascertain whether, consistently with the above, any *new* epistemological principles emerge. And this means in particular the investigation of the doctrine of so-called 'necessary truth', with which we have not so far met. (This will no doubt emphasize the essentially arbitrary nature of our distinction between common and advanced knowledge. For the mathematical proposition $2+2 = 4$ has been supposed to possess 'necessary truth', and because it is mathematical, we include it in the category of advanced knowledge. Yet I suppose it is known to most savages.) We shall finally (3) attempt to incorporate all such principles, both those already discovered in our analysis of common knowledge and those which may yet emerge, into a single comprehensive theory of knowledge.

For reasons of convenience I shall begin with the study of mathematical knowledge, with special reference to geometry.

Our inquiries have shown us that knowledge is not entirely free. In spite of the 'will to believe' we cannot believe whatever we please. Knowledge is tied down at its lower end. It is tied to the given. And we have now to inquire whether it is tied at its upper end also. The given compels me to think thus and thus. I cannot think 'This is red' if the greenness of the 'this' stares me in the face. Is there any similar compulsion in the conceptual sphere? Does the concept itself ever compel me to think thus and thus? The assertion that it does is the essence of the theory of 'necessary truth'.

When we say 'This is red' we are stating a mere fact. No doubt we are under compulsion to think this fact. But the compulsion comes from the fact itself, not from the mind. It is thought of as a compulsion which comes from *without*, from the external world. Consequently there appears to be nothing within thought which is necessary. That the object before me is red is considered a mere contingency. It might well have been otherwise. It might have been blue. It would not have been possible to predict, without looking at it, that it would be red because it *must* be.

But the thought that $2+2=4$ is supposed to be *necessary*. It is not a mere fact like the redness of the object before me. Two plus two not only *is* four. It *must* be. It could not be anything else. It is conceivable that the object before me might have been blue. But it is inconceivable and impossible that two and two should ever be five. In the same way geometrical truths, such as the axioms of Euclid, were at one time supposed to be necessary truths.

The doctrine of necessary truth has exercised a profound influence in philosophy, and every epistemology must needs examine it. It goes back at least to the time of Plato. In the *Meno* Socrates is represented as showing geometrical figures to a slave and, by means of skilful questions, eliciting from the slave various propositions of geometry. The point is that Socrates *tells* the slave nothing. He only asks questions. It is assumed too that the slave is wholly ignorant of geometry. Yet being shown the figures, and being asked the right questions about them, he enunciates geometrical truths. Since he was neither told them by Socrates nor knew them before, where did he learn them? It appears from this that the mind can somehow produce geometrical knowledge out of itself. And Socrates concludes that this knowledge must have been remembered from a previous life in another body, and bases thereon his belief in the doctrine of reincarnation.

No doubt it would appear strange to us nowadays to base an argument for immortality or reincarnation on geometry. But that is not the point. What I wish to emphasize here is that we have, in this passage of the *Meno*, a clear anticipation of the doctrine of necessary truth. Plato has seen that there appears to be a kind of knowledge which is not derived from experience, which is, in some way, prior to experience. This is, so far as I know, the earliest hint of the doctrine of *a priori* knowledge which has played so large a part in some modern philosophical theories. Plato based upon it the doctrine of reincarnation. Modern theorists have built upon it a species of transcendentalism, a belief in the existence of another world beyond experience, which, if true, would be no less profoundly important.

It was Kant who, in modern times, was responsible for this. Hume had shown that necessity cannot be derived from experience. Experience can only prove that a thing *is*, never that it *must* be. We see that something is green. No amount of staring, no amount of examination by a microscope, can ever reveal any 'must be' in the experience. Nor will the multiplication of facts or experiments alter the case. You may pile facts upon facts for ever, but they still only 'are'.

But in geometry, Kant thought, we know not merely that a proposition is true, but that it is necessarily so. Since this knowledge cannot be derived from experience, it must be imposed upon experience by the mind. And since Kant believed that geometry gives us knowledge of space, he argued that space is ideal, a form of our perception which is not in things themselves, but which our minds create as a framework into which things have to fit themselves before they can enter into our knowledge.

Since he believed that necessity also attaches to the categories, he drew the same conclusion as regards them. They too constitute a framework which the mind imposes upon nature. Through this gateway there entered into modern philosophy the doctrine of a universal cosmic mind. And upon this depended the whole of that kind of transcendental idealism which dominated European philosophy from the time of Kant till about thirty years ago. So great has been the influence of the doctrine of necessary truth.

Necessary truth has been attributed to

(1) The propositions of geometry and mathematics generally.
(2) Categorial knowledge.
(3) Logical knowledge.

I shall in this chapter investigate the first of these heads, leaving the others for later examination.

Geometry is a system of truths which follows by logical necessity from a set of axioms, postulates, and definitions. Definitions are admittedly merely analytical propositions, i.e. propositions in which the predicate is an analysis or partial analysis of the subject, so that what they state is merely the meanings of terms. With the postulates we need not concern ourselves. It is of the axioms mostly that we shall have to speak. According to the older views, they were regarded as necessary or self-evident truths. They were incapable of proof and did not need it. The guarantee of this truth was their intuitively perceived necessity. It *must* be true, so it was thought, that two straight lines cannot enclose a space. This self-evident character of the axioms rendered them fit to be the ultimate foundations, the ultimate premisses, of geometry. It was upon this basis chiefly that Kant reared his doctrine that space is not an ultimate reality.

Kant of course had other arguments by which he sought to prove the ideality of space, but with these we are not concerned. And as for this argument, the whole basis of it has been completely undermined by two discoveries: (1) that certain of the axioms of Euclid are not self-evident, nor necessarily true, but are pure assumptions for the truth of which there is no guarantee; and (2) that those axioms which are not pure assumptions are disguised definitions, or analytic propositions.

The first of these discoveries is the result of non-Euclidean geometry. There seems to be an idea in some quarters that it is a consequence of Einstein's theory of relativity. This is a mistake. The theory of relativity has no direct bearing upon the views of space and of necessary truth which we are discussing. Indirectly it has had an influence by bringing non-Euclidean geometry into the limelight of popular discussion. But non-Euclidean geometries had been known to mathematicians for nearly a century before Einstein's theory was propounded. The layman might, however, have continued to regard them as mathematical curiosities, mere puzzles remote from reality, had it not been for the fact that Einstein has insisted that the space in the neighbourhood of the sun and other gravitating masses actually *is* non-Euclidean, and that this has to be taken account of in explaining such concrete matters as gravitation and the orbits of the planets. All this has forced non-Euclidean geometry upon the popular imagination. But apart from this Einstein's physics has absolutely nothing to do with the questions we are discussing.

The discovery of non-Euclidean geometry arose chiefly from reflection upon Euclid's axiom of parallels. That axiom may be stated in the following form: *If l be any straight line and p any point outside l, then there is one and only one straight line through p, and in the plane which contains p and l, such that it does not intersect l.*[1] Most of the axioms, it was thought, might be self-evident. But this axiom of parallels certainly is not. Generations of mathematicians, therefore, tortured their brains to madness in the effort to find a proof of this proposition. All attempts failed. And it was therefore suspected that the axiom is neither self-evident nor capable of proof by deduction from any simpler or more fundamental axiom, but that it is logically independent of the other axioms. This led Lobachevsky and Bolyai, working independently of each other, to proceed on the assumption that perhaps it might be legitimate to regard the axiom of parallels as untrue, and to suppose that some other hypothesis about non-intersecting straight lines might be true instead of it.

The geometry of Lobachevsky if founded on the assumption that through the point *p more than one straight line may be drawn such that it will not intersect l.* At the same time Lobachevsky adopts all the other axioms of Euclid. From these foundations he proceeds to deduce as logical consequences a number of theorems which constitute the body of his geometry. In this geometry, of course, many

[1] I am indebted to the kindness of Dr. C. D. Broad for this way of expressing the axiom.

of the theorems differ from those of Euclid. For example, in Euclid's geometry the sum of the three angles of a triangle is equal to two right angles. But in Lobachevsky's geometry the sum of these angles is always *less* than two right angles. And many other consequences follow which from a Euclidean point of view we should regard as very strange.

On this question of pairs of non-intersecting straight lines there are three logical possibilities and only three. (1) Through the point *p* there is *one and only one* straight line which does not intersect *l*. This is the assumption made by Euclid. (2) Through the point *p* there is *more than one* straight line which does not intersect *l*. This is the assumption adopted by Lobachevsky. The third logically possible alternative is that (3) through the point *p* there are *no* straight lines which do not intersect *l*. This third assumption is that upon which Riemann built his geometry.[1] This geometry is different from both Euclid's and Lobachevsky's. In Riemannian geometry, for example, two straight lines can enclose a space, and the sum of the three angles of a triangle is greater than two right angles.

The geometries of Lobachevsky and Riemann are just as internally self-consistent as that of Euclid. And their foundations are just as sure. For if the new axioms on which they are built are neither self-evident nor capable of proof, exactly the same can be said of Euclid's axiom. Between the three geometries there is, so far as *internal* evidence goes, nothing to choose.

Whether one or other can be established by *external* empirical evidence, for example by measuring the angles of some huge triangle in stellar space and seeing whether they are more than, equal to, or less than two right angles, is another question, to which I shall revert. But the present point is that it is proved that the axiom of parallels is not a self-evident or necessary truth at all, but a pure assumption.

Whoever is biased in favour of Euclid may perhaps

[1] I am also indebted to Dr. C. D. Broad for this way of exhibiting the logical relations of the three geometries to one another.

attempt to retain one hope. The axiom of parallels is not self-evident and has not been proved in the past. But is it not perhaps still possible that mathematicians may yet find a proof? Perhaps it may be one of those mathematical problems which may be believed to be soluble though not yet solved. Vain hope! For Beltrami has proved that to prove the doubtful axioms is logically impossible.

If one or the other set of axioms can be established by astronomical measurements or other empirical evidence, then they are propositions founded on experience. In that case they possess no more necessity than any other statement of observed fact. If no such experimental proof is possible, then the axioms under discussion, to whichever geometry they belong, are pure assumptions. It is then entirely a matter of convenience which geometry we choose to adopt. In that case too they clearly cannot be regarded as necessary truths.

But what about the *other* axioms of Euclid, those which no one has ever doubted, those which are common to all the different geometries? Are not they at least universal and necessary truths? The Kantian might still attempt to found his argument for the ideality of space on that ground. But unfortunately for him this position too is untenable. For those axioms of Euclid which are not unprovable assumptions like the axiom of parallels are disguised definitions. They are no doubt universally and necessarily true, but only because they are analytic propositions which state nothing more than the meanings of terms.

As an example of an axiom which is merely a disguised definition take the following: 'Magnitudes which can be made to coincide with one another are equal.' It is easy to see that this is nothing but a definition of what we mean by the term 'equality of magnitudes'. A rod *AB* is called equal to a rod *CD* when we can pick them up, put them together, and find that the ends coincide. This is what we *mean* by calling them equal. And this meaning of the term equality is all that is stated in the axiom.

The axiom 'The whole is greater than its part' is a partial and incomplete definition of the part-whole relation. Whatever the complete definition of that relation may be, it will certainly imply and include the fact that the whole is greater than its part.

Definitions are analytic propositions. They merely state the meanings of words. 'All horses are animals' is an analytic proposition. 'Animal' is, of course, part of the meaning of the word 'horse'. Analytic propositions are eternally true. They are universally and necessarily true because they make no statement about outward facts, but only express our decisions as to how we intend to use our terms. A horse must always necessarily be an animal for the simple reason that animal is part of the meaning of the word horse, and if any object placed before us were not an animal we should not admit that it could properly be called a horse. The universality and necessity of those Euclidean axioms which are not pure assumptions is of exactly the same character. If two rods will not coincide we shall refuse to admit that the word 'equal' can be used of them. It must be eternally true that if P is a whole and p a part of it, then P must be greater than p, because, if not, we should not call P a whole and p a part of it.

But the whole point of Kant's argument was that the axioms are synthetic propositions. He thought that they stated real truths about space, truths which no mere analysis of concepts or knowledge of the meanings of terms could yield. 'This horse is lame' is a synthetic proposition. You cannot discover its truth by analysing the meaning of the word 'horse'. Lameness is not a part of the definition of a horse. You can only discover that this horse is lame by examining it.

If it were really the case that we could, as Kant thought, assert synthetic propositions about space which yet had the property of necessity, this would certainly be a most remarkable fact. We can understand why an analytic proposition is universal and necessary, namely because it is tautologous. But how can universality and necessity attach to synthetic propositions? They cannot be gathered from experience, and since synthetic propositions are not tautologous their necessity is not lent to them by logic. Kant found no way of explaining this except by assuming that space itself, regarding which he supposed that we have such necessary synthetic knowledge, is a product of our own minds. If the axioms were synthetic propositions, we might have to explain their necessity by some such far-reaching metaphysical hypothesis. But they are not. Those which are not pure assumptions are definitions or analytic propositions, and their universality and necessity are quite simply explained without having recourse to the speculation that space is unreal.

Our present aim, however, is not to discuss the nature of space or Kant's views thereon, but to decide whether knowledge, as well as being tied at its lower end by the given, is also tied at its higher end by the necessity of its own concepts. We wanted to sift the doctrine of necessary truth as it is alleged to appear in mathematics, in categorical knowledge, and in logic. The result so far attained is that geometry does indeed contain necessary truths, but that they are purely analytic. This means that their necessity is not peculiar to mathematics or to geometry. It is exactly on the same footing as the necessary truth of the propositions 'All horses are animals' and 'All unicorns have one horn'. In other words it is not a mathematical necessity at all but a purely logical necessity.

You cannot admit the truth of a proposition the predicate of which contradicts the subject. 'This unicorn has two horns' *must* be false because the word unicorn *means* a particular kind of animal with one horn. You *must* admit the truth of a proposition the predicate of which is included in its subject-concept such as 'horses are animals'. Why *must* you? Obviously because to do otherwise would be self-contradictory. And the necessity not to contradict onself is a law of logic, not a law of mathematics.

This is a very important result. It does not mean that there is no necessary truth in geometry. But it does mean that the source of this necessity is pushed back, out of the sphere of mathematics, into that of pure logic.[1] The truths of geometry are not *in themselves* necessary. They are deductions from logical laws. We shall therefore be compelled, in attempting to solve the problem whether knowledge is tied at its conceptual end, to carry our study into the sphere of pure logic. We shall do so in the chapter on logical knowledge.

What has been proved in detail of geometry is equally true of arithmetic and other branches of mathematics. The proposition $2+2=4$ is necessarily true only because it is analytic. The necessity of mathematics generally, then, will have to be studied under the head of logic.

It may be asked whether there is any universal and necessary knowledge of time, and the irreversibility of time may be given as an example. But the opposite of this, namely the reversibility of time, is inconceivable. Suppose that all natural processes were reversed; that grey-haired old men grew younger and returned to the cradle; that oak trees retreated slowly into acorns. This, it is surely obvious, would not be the reversal of the time-order. The acorn would then come *after* the oak instead of before it. Its date would be later instead of earlier. The reversibility of time itself would mean that a later moment of time must become an earlier moment. But this is merely a contradiction in terms. To speak of the reversibility of time is precisely like speaking of the blackness of white. The truth that time is irreversible is thus no doubt universal and necessary. But the proposition is analytic. Irreversibility is part of the concept of time. 'Black is not white' is a necessary truth. So is the denial of any other self-contradiction. But all such denials are implied by the law of contradiction, and their necessity is lent to them by logic.

[1] It is true that mathematical philosophers now tend to disregard the boundary, and regard the two spheres as indistinguishable. Even if this is admitted, it does not traverse my point, which is that the necessity of mathematics is simply a general logical necessity. Indeed it strengthens that point.

We have investigated the supposed necessity of mathematical truth up to the point at which it disappears out of the territory of mathematics into that of logic. We have decided to pursue it over the boundary in a later chapter. But in the meanwhile we must face another problem. Apart from the question whether mathematical propositions are necessarily true, what is the meaning of saying that they are true at all? What is the nature of mathematical truth? And has the answer to this question any bearing upon the general problem of truth which confronts epistemology?

Let us begin with the simplest kind of arithmetic. What is the meaning of saying that the proposition $2+2=4$ is true, and that the proposition $2+2=5$ is false? I cannot doubt that this knowledge, like all the knowledge we have so far studied, is tied by the given. In other words these propositions refer to concrete facts. To say that $2+2=4$ is true because, if you take two things and place them in a group with two other things, then the whole group will be four things. The proposition is true because it agrees with the empirical facts. The proposition $2+2=5$ is false because it is contradicted by the facts. It is of course true, as so often stated, that mathematics is about numbers, and not about pigs and horses or any other particular things. In the same way the law of gravitation is about masses in general, not about the earth, the moon, or any other particular mass. But the truth of the law of gravitation resides in its applicability to the earth, the moon, &c. Arithmetical laws are similarly general and apply to all numerable things. But their truth must reside in their applicability to particular things. No doubt the mathematician is only interested in the concept of pure number. But, unless that concept had application to the real world, mathematical propositions could not be described as either true or false.

Exactly the same must, in my opinion, be said of geometry. But there are several difficult questions involved here, which it will be necessary for us to discuss.

Physicists are busy discussing whether space is Euclidean or non-Euclidean, or whether it is Euclidean in some places and non-Euclidean in others. What we are here concerned to ascertain is, not which of these statements is true, but what is meant by saying that any of them is true. To say that space is or is not Euclidean seems clearly to imply that Euclidean geometry applies or does not apply to space. It implies that *some* geometry applies to space, and therefore that some geometry—whether that of Euclid, Riemann, or Lobachevsky—is true. What is meant by the statement that a geometry is true?

There appears to be a tendency in some quarters to identify its truth with its internal self-consistency, and to assert that the ascription of truth to it in any other sense is unmeaning. Mathematicians are fond of picturing themselves as existing in a world of their own, a world of abstract symbols, completely aloof and cut off from all concrete reality. They are superior to mere 'things' and haughtily decline to know anything about them. Mr. Bertrand Russell tells us that 'mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true'. And I once knew a mathematician whose hobby was gardening, and who, throughout an entire hour's lecture on the irregular polyhedron, referred to it absent-mindedly as the 'irregular rhododendron'.

That pleasant soul the mathematician in the dialogue at the beginning of Professor Eddington's *Space, Time, and Gravitation*, expresses himself as completely unconcerned whether the axioms of geometry are true or not. For him they are simply propositions from which he will deduce the logical consequences. He is equally prepared to deduce the consequences of any other set of propositions. *Which* set of propositions he works with, and whether they are true or false, is to him a matter of indifference.

This rather tall talk of the mathematicians fulfils its function in the world by expressing a pleasant and amusing pose. But it will mislead us if we take it too seriously. It is regrettable to have to point out to the mathematician that arithmetic does in the end apply to pigs and tons of coal, and that the theorem of Pythagoras is concerned with the measurement of solid bodies. But such is the truth.

No branch of knowledge exists cut off and alone in a universe of its own. It is surely a platitude that knowledge is one, and that every part of it stands in some definite relation to every other part. And mathematics cannot be an exception. It must stand in relation to the real world of concretes. A system of geometry must be either true or false. And its truth cannot consist in the mere fact of its being internally self-consistent.

It is pointed out by those who wish to keep geometry free from contamination of the world that it is not necessarily—as Kant supposed—the science of space, and that it has not even any essential connexion with lines, angles, and points. It can be so generalized by the use of symbols that the symbol may stand for a line or it may stand for *anything*. So that the truth of the system is independent of whether there are such things as lines, points, &c., or not. Geometry then becomes an exercise in pure logic. A reader of it might understand completely the whole of such a system of geometry without even knowing what a line or a point is.

But this argument is irrelevant. No doubt geometry can be generalized till it ceases to be geometry and becomes some kind of mathematical logic. In the same way arithmetic can be generalized till it becomes algebra. But it is still true in spite of this that the proposition $2+2=4$ is a law which applies to pigs, horses, and cabbages. Nor is this truth embarrassed by the mathematician's irrational and imaginary numbers. And it is still true, in spite of the generalization of geometry into something more abstract, that the theorem of Pythagoras (whether in its ordinary or in some more remote and abstract form) can be applied to, and is true or false, in some sense yet to be defined, of plots of land and other material things. The generalization of geometry may be important, both practically as the development of a more powerful mathematical instrument, and theoretically as showing the dependence of mathematics on logic. But it has no bearing on the problem with which we are at present concerned, the problem of the truth of geometry.

Neither need we be alarmed when we are told that geometry can be made, so to speak, to stand on its head. We can begin, it appears, with the proposition *p* (which may be one of Euclid's axioms) and deduce from it the theorem *q*. The order of propositions '*p* therefore *q*' is the order adopted by Euclid. But we can reverse this order without in any way injuring the validity of the mathematical structure. We can begin with the proposition *q*, treat it as our initial assumption or axiom, and deduce *p* from it as a theorem. This possibility may appear to favour the complete independence and self-sufficiency of geometry. But it does not. The two ends of the system, *p* and *q*, must both *mean* something. They must have some application to the real world, and they must make some true or false statement regarding it. Whether we begin with *p* and end with *q*, or vice versa, is a matter of complete indifference. The fact that two propositions are mutually deducible from one another has no bearing upon the question of what is meant by the truth of the propositions.

The discussion whether space is Euclidean or non-Euclidean implies that some geometry applies to, and is true of, space. This again implies that geometries have a reference to a reality outside themselves, and are not wholly self-enclosed. It does not necessarily imply, however, that if one geometry is true the others must be false. And I shall in fact argue that *all* the admittedly self-consistent geometries are true; and that which one we choose to adopt in any case is a question, not of truth, but of convenience. I shall return to this point later.

If then a geometry, whether Euclidean or non-Euclidean, must claim to be in some sense true of the real world, we must now go on to inquire how this truth is to be understood. *What* is meant by saying that a geometry is true? The question presents difficulties at the outset because, as has often been pointed out, geometry deals with apparently non-existent objects. There is no such thing in the world as a line, i.e. an object which has length but no breadth. Neither do such things as circles, planes, and points exist. What corresponds to a line in nature is the edge of the leaf of the plantain-tree which stands opposite my window. What corresponds to the straight line is the edge of my desk. What corresponds to a circle is the wheel of my motor-car. But none of these things are the perfect objects imagined by geometry. The edge of the plantain-tree is vague and indefinite, and must be regarded as having at least *some* width. The edge of my desk is not perfectly straight, as the microscope or even the naked eye will show. The wheel of my car is not perfectly circular, and in any case is not a mathematical line with no breadth.

Perhaps for this reason there arose the idea, favoured by Kant and others, that geometry is the science, not of rigid solids, but of pure empty space. There are no material things which can be said to be circles, lines, or points. But these objects, it may be thought, exist in pure space. Geometry in speaking of a straight line refers, not to the edge of my desk, but to a line between two points in empty space.

For my part, however, I am unable to discover any lines, circles, or points in space. When I look out into space I see either a material object or absolutely nothing. Possibly this is due to the fact that my sight is poor and that I have to wear spectacles. But I think it more likely that it is the same with all of us, and that we cannot see lines and circles in space because they are not there to see. Lines and circles are not *given*. Nor are they valid inferences from anything that is given. They are abstractions. They arise by the mind abstracting from the characters of real things and their relations. So that in the last resort geometry must apply, not to space, but to material things.

Now let us in the light of this supposition endeavour to interpret in terms of actual things some simple proposition of geometry. We will take for this purpose the fourth proposition of the first book of Euclid. This theorem states that if two triangles have two sides and the contained angle equal, then the triangles are equal. The proof depends upon the axiom that 'magnitudes which can be made to coincide with one another are equal'.

It is obvious that if this proposition is intended to apply to triangles in pure space it can have no meaning. You cannot move one part of empty space from where it is and superimpose it upon another. And if there were such things as lines and triangles in empty space they likewise could not be moved. In order to give the theorem any meaning we have to think of actual things. If, for example, we cut out two triangular pieces of metal which approximately satisfy the conditions laid down by Euclid about the angles and sides, then if we put one on top of the other we shall find that these objects roughly coincide with one another. That is the meaning of Euclid's fourth proposition. And the truth of the proposition consists in its correspondence with the empirical facts.

But it may be objected that statements about pieces of metal and the like are not what is found in books of geometry. They speak of perfect pure triangles, circles, &c. If material things were perfectly circular, triangular, and so on, then we might say that geometry applied to them. But they are not. What exists or does not exist is, of course, entirely a matter of empirical evidence. And observation easily decides that there are nowhere in the world any pieces of metal having three sides which are straight lines and such that their angles and sides absolutely coincide.

Geometry is, so far as we see at present, in exactly the same position as a science which should take as its fundamental axiom or assumption the proposition 'Unicorns are one-horned horses'. From this we might deduce the theorems 'Unicorns have four legs', 'unicorns are mammals', 'unicorns have two eyes', and the like. Such a science, if called true at all, must be called universally and necessarily true. For the axiom on which it depends is a definition or analytic proposition, and all the theorems deduced from that axiom are equally analytic propositions.

But what such a science could not from its own resources tell us would be whether there are in the actual world any unicorns. To answer that question we must have recourse to empirical observation.

Geometry seems to be just such a science, and observation tells us that there are in the world no such things as pure lines and triangles, and no such things as perfectly triangular or circular pieces of matter. What then is the application of geometry to things, and wherein does the truth of the science lie?

Clearly, though there are no exact triangles and circles, there do exist objects which can, for practical purposes, be regarded as roughly triangular, circular, &c. And the truth of geometry must have something to do with this. My desk-top is roughly a rectangle. I could superimpose on it another desk-top such that the edges and corners would roughly coincide, and I should then call them equal in size. And the diagonal of my desk-top if measured will be found to be such that the square on it is roughly equal to the sum of the squares on the two unequal edges. What the theorem of Pythagoras seems to assert is that this will always be true in all similar cases.

But there is still a gap between the ideal truth of geometry and the empirical truths about the measurements of real things which correspond to them. How is this to be bridged?

It does not appear to be a problem difficult of solution how the mind constructs the concepts of ideal figures. The mind having begun a certain process finds nothing to stop it in its continuation of the same process to the ideal limit. We have, for example, empirical knowledge of elastic bodies. We can arrange them in a series of increasing elasticity. The mind can then continue the series beyond what is actually given in experience. It arrives ultimately at the concept of perfect elasticity, although no perfectly elastic body is ever found in experience. In the same way the mind creates the ideas of infinite space and of the mathematical continuum. Bodies are extended. There is no logical contradiction in mentally carrying the extension beyond the bodies outwards indefinitely. Hence the concept of infinite space. Again, the mind finds that between the members of a series it can always insert intermediate members. There is no reason why this process should ever stop. Hence the idea of the mathematical continuum. The method of constructing geometrical figures is no different. We find in experience strips or bands of material which we can arrange in a series of decreasing width, while the length remains the same. We have only to continue this process in thought to its ideal limit to arrive at the concept of the geometrical line.

Consider the formula $v = gt$ for the acceleration of bodies falling in an absolute vacuum. We have no experience of an absolute vacuum. Experiments can be made in the normal atmosphere or in any partial vacuum which our physical apparatus can produce. And it is found that the nearer we approach to an absolute vacuum the more nearly it is true that $v = gt$. This formula is therefore set up as an ideal limit.

Now in all cases where it is possible to construct an ideal limit towards which empirical facts may approximate, there result two consequences. (1) The statement of the ideal limit being taken as fundamental premiss or axiom, there may be deduced from it a series of propositions which will have the characters of a deductive science. Provided we admit the truth of the fundamental axiom (i.e. provided we ignore its ideality or variation from the real) the propositions of the science will be universally and necessarily true. (2) This science will be actually true of reality in the same degree as its original premiss is true of reality. That is to say, the nearer real things approximate to the ideal limit imagined in the original axiom, the more nearly will all the propositions of the science be true of real things.

We can deduce from the concept of ideal elasticity propositions which are true of real elastic bodies in the degree in which they approach to ideal elasticity. From the formula $v = gt$ we can deduce results which become progressively truer as real conditions approach a perfect vacuum.

Geometry is a science of this kind. The ideal limits which we set up are straight lines, circles, triangles, &c. We deduce from these assumptions a set of propositions which are rigorously and necessarily true *if* the assumptions are true, i.e. if we ignore their ideal and abstract character. And the more nearly my desk-top approaches to an ideal rectangle, the more nearly will the theorem of Pythagoras be true of it. This explains (1) why geometry, if taken in abstraction from reality, is an exact deductive science possessed of rigorous certainty, and (2) in what sense it is nevertheless true of empirical reality.

Jurisprudence offers not unhelpful parallels. For example, the definition of contract is likely to include the conception of the agreement of two minds. And the law lays it down that if there is an agreement between two minds of the specified kind, then certain rights and duties will arise. Given this assumption the law becomes in some degree a deductive science. In point of fact, however, there is no such thing as a perfect agreement, since two minds, however well attuned, always misunderstand each other in some measure. But observation of the moral and business customs of men shows that where there is a rough agreement of minds, rights and duties come into existence. Where the divergence of minds is so great as to amount to what is called mistake, no obligations arise. Between the two extremes there may be many intermediate possibilities. And common sense would conclude that the nearer the agreement is to perfection the more decidedly ought the rights and duties to be enforced. An ideal limit—perfect agreement—is thus conceived and expressed in the form of a definition, from which rigorous deductions can be made. The definition here corresponds to the axioms of geometry which are, as we have seen, disguised definitions.

One might without difficulty construct a science of the ideal man. Man would be *defined* as perfectly rational, wise, just, humane, moral, artistic, and so on. From this one would conclude that men in given circumstances would do thus and thus. Such a science would be in the same position as regards reality as geometry. Just as we argue that, because in perfect geometrical circles the relation of the circumference to the radius is $2\pi r$, therefore this relation in the case of the wheel of my motor-car will be somewhere round about that; so we might argue that since a perfectly just man would do thus and thus, therefore Smith, who is well known to be just as men go, will actually do so and so.

Our conclusion is that the truth of geometry consists in its empirical application to real things. But it may well seem that difficulties are raised for such an opinion by the existence of the non-Euclidean geometries. For it might be argued as follows. If the truth of a geometry is thus its application to the real, then it is clear that we must decide *which* of the rival geometries is true and which false by seeing which applies to the facts of the real world. This could only be done by measurement. We should have to measure the angles of some vast interstellar triangle, and see which geometry they fit. But there is good reason to believe that such a course is fundamentally fallacious, and that no such measurement could possibly decide between the geometries. Therefore the question of which geometry is true cannot depend on which fits the empirical facts. And in that case it is difficult to believe that truth in geometry means what we have stated it to mean.

Let us first get clear as to why astronomical measurements cannot decide between the different geometries. Suppose we measure an astronomical triangle, and find that the sum of its angles is equal to two right angles. We draw a conclusion in favour of Euclid. If we find the sum of the angles less or more than two right angles we conclude in favour of Lobachevsky or Riemann. What is wrong with this argument?

We may ignore the practical difficulties of measurement and the inaccuracies of human methods and instruments.

We may assume that a perfectly accurate measurement has been taken. If then we find that the sum of the angles exceeds two right angles, cannot we deduce that the geometry of Riemann is the true one? We can if we like, but this conclusion will be based on the assumption that *light travels in straight lines.* We can also explain the measurement of the angles which we have made on the opposite assumption, namely that light travels in curves and that the geometry of Euclid is true. There is no good reason whatever for supposing that light follows a straight path, except that this is a convenient basis for optics. But the laws of optics could equally well be worked out on the assumption that light travels in curves, except that the expression of these laws would be more cumbrous. Thus, if we make the astronomical measurement imagined above, it is a pure matter of convenience whether we keep the ordinary laws of optics and alter our geometry to that of Riemann, or whether we keep to Euclidean geometry and alter the laws of optics. We can choose whichever course we like, and the one which we shall actually choose will of course be the one which is simplest to work.

What then is meant in the theory of relativity when it is stated that space in the neighbourhood of gravitating masses *is* non-Euclidean, and that the famous observations of stellar displacements made at eclipses of the sun definitely support this view? What is meant is simply that the easiest and most convenient way of explaining the displacement of the star is to assume that light travels in straight lines and that a non-Euclidean geometry is true. By altering various other laws and conventions of science we can explain all the facts on which the theory of relativity and Einstein's law of gravitation is based on the Euclidean hypothesis. The whole theory of relativity can be expressed in terms of Euclidean geometry. But to explain the facts in this way would be vastly more complicated.

Thus we see that no measurements can prove that one geometry is truer than another. Such experiments only show which is the most convenient. And it was for such reasons that Poincaré expressed the opinion that to ask whether a geometry is true is meaningless. Geometries, he said, are not true at all, but only convenient; and some are more convenient than others. And this may well appear to be in contradiction to our view that geometry has truth and that this truth consists in its applicability to concrete things.

The contradiction, however, is only apparent. We can admit the truth of Poincaré's contention that the choice between geometries is decided by nothing more than convenience, but stick to our own opinion that geometries have truth in that they apply to the real world. In other words, *all the three geometries are true*, since they can all be applied to concrete reality. But the choice between them does not depend on any difference in their truth but on differences in their relative convenience in use. It is certainly nonsense to speak of one geometry as *truer* than another. From this Poincaré seems to have concluded that no geometries are true. I would conclude, on the contrary, that all geometries are equally true.

We can see this if we will consider some of the illustrations of his views with which Poincaré is so lavish. He compared the different geometries to different languages. Just as it is purely a matter of convenience whether you say what you have to say in English, in French, or in German, so it is equally a matter of convenience whether you use Euclidean or non-Euclidean geometry. Or again, to ask whether the geometry of Euclid or that of Lobachevsky is true is like inquiring whether the truth about the temperature is given by the Centigrade or the Fahrenheit thermometer; or whether a distance is more correctly measured in yards or in metres. It is entirely conventional on which scale we measure temperatures or by what units we measure distance. It is equally conventional which of the geometries we use.

This is all quite true. But these very illustrations of Poincaré's imply, not that no geometry is true, but on the contrary that all geometries are true, that they all express the same truth in different ways. You can express the same truth in English, French, or German. It is a mere matter of convenience which language we use. But this fact does not imply that none of the languages can tell the truth. Fahrenheit and Centigrade thermometers both give truth about the temperature. Measurements of a distance in yards and in metres may both be accurate.

Thus we come back to the conclusion that mathematical knowledge is about 'things', and is tied to the given in the same way as other knowledge already investigated on earlier pages. This does not, of course, enable us as yet to establish a general theory of the nature either of mathematical or any other truth. It establishes only one preliminary point in the theory. It shows that mathematical knowledge does not differ from other varieties of knowledge, except in the fact that it is more abstract. All knowledge, whether it is comparatively concrete or comparatively abstract, refers in the end to sensible reality, and takes its meaning and its truth from that reference. No knowledge can, like a balloon, cut its moorings to the earth and rise into empty space.

It has long been recognized by mathematicians and logicians that the three axioms regarding non-intersecting pairs of straight lines which lie at the basis respectively of the three geometries are not self-evident and necessary truths, but pure assumptions. But how and why the mind can make such assumptions, what its justification for doing so is, and what is the general position of such assumptions in the scheme of human knowledge—on these questions, so far as I know, mathematicians and logicians have had nothing helpful to say. These geometrical assumptions, proved by nothing, founded on nothing, not self-evident, hanging in the air, and yet the foundations of the rest of geometry, have appeared as mere mysteries. Mathematics is a mysterious science, ruled over by magicians. They, of course, can do anything. It is quite right that they should place at the base of their science unprovable assumptions which have descended to them apparently straight out of the empyrean. And this wand-waving mystery business has been skilfully cultivated so as to impress the simple-minded. It was supposed that it was *only* mathematics which could behave in this queer way. No other science or branch of knowledge could take as its fundamental premisses unproved assumptions. Biology, chemistry, geography, history, could not do it. Certainly common sense could not do it. These geometrical axioms, therefore, were thought of as unique in knowledge.

But now it would seem that we can at last bring these axioms into line with other knowledge. For our investigations throughout this book have shown that they are not unique, that, in fact all knowledge is based upon just such unprovable assumptions. And in this way we can bring geometry with its axioms into general epistemological theory, and not leave it standing outside, a mysterious exception to all rules. For there is no essential difference between the logical position of such a proposition as that which asserts that the table exists when no one is looking at it and the logical position of the three axioms regarding non-intersecting straight lines. These axioms are just as much mental constructions as the proposition about the table. For they possess the essential character of all mental constructions, namely that they cannot be proved by any conceivable means. They are not given. They cannot be inferred from anything that is given. They are creations of the mind.

To what type of construction do these geometrical axioms belong? Are they unificatory or existential? They certainly are not unificatory, for there is no sense in which it could be said that they abolish superfluous existences or reduce many existences to one. And at first sight it may appear difficult to recognize them as existential constructions. For the existential construction to which we have so far been accustomed always creates in imagination a new sense-object, or at least a new relation between sense-objects. The table when no one is aware of it is a sense-object conceived as existing outside actual perception. The resemblance between your red and my red, which was the first construction of Chapter VI, was the creation of an unperceived relation between two sense-data. The geometrical axioms do not appear in this way to assume the existence of sense-data or their relations outside the sphere of perception. But a closer examination reveals that the apparent difference is merely due to the more abstract character of geometry. The geometrical axioms create, not sense-objects outside perception, but geometrical objects. For the existences with which they are concerned are the intersections or non-intersections of straight lines in an unperceived extension of space.

Any space which we actually perceive (or imagine) must be a limited space. For the sake of simplicity we will think in terms only of visual space. Any actually perceived visual space is of course limited by the boundaries of the field of vision. We perceive (1) that many straight lines within this space intersect, and (2) that many pairs of straight lines reach the boundaries of the space without intersecting. Our previous constructions have taught us that space continues indefinitely beyond the limits of our perception. And therefore as regards the pairs of straight lines which are not seen to intersect within the visible space the question arises whether they will intersect if they are extended beyond it. Some of them undoubtedly will. But will they all? Will there be any pairs which, however far they are produced, will never intersect?

It is to this question that Euclid, Lobachevsky, and Riemann give different answers. Riemann's assumption is that *all* the straight lines will intersect, that there are no pairs of non-intersecting straight lines. Euclid's assumption is that if you take a given straight line *l* there is, through a given point outside it, one and only one straight line in the same plane which will not intersect *l*. Lobachevsky's assumption is that there is more than one such straight line.

The constructions are therefore existential because they deal with the question of the existence or non-existence of points of intersection outside the field of perception. Riemann asserts that in all cases such points of intersection exist. Euclid makes one exception, Lobachevsky more than one. Clearly then what these assertions mean is 'If you could perceive the continuation of straight lines outside the limits of perception, you would perceive the existence of points of intersection in all cases, in all cases except one, in all cases except more than one'. The existence constructed is the point of intersection outside experience. And we have, as usual, the 'if' clause expressing an impossible condition.

Since, as we have seen, all the three geometries are true, they constitute, therefore, an interesting example of the possibility in knowledge generally of *alternative truths*. We have previously had instances of such alternatives. We saw that certain changes in the visual field, for example the appearance of a white billiard ball moving across the green cloth of a billiard table, might be explained in two ways. We might believe either (1) that visual space is two-dimensional and that the observed change is no more than a change of state, to wit, a change of colour. Or we might believe, on the other hand, (2) that that visual space has three dimensions, that empty space exists, and that the motion of solid bodies such as billiard balls takes place in this empty space. The mind might have adopted either of these explanations. There is nothing to prove one as against the other. Even to-day if any mind wishes to adopt the first alternative there is nothing to prevent him doing so, nor anything to prove him wrong. But he would have to face the difficulties which the human mind in general faced when it came in the course of its evolution to this particular parting of the ways. The mind actually chose the second alternative because the first would have been inconsistent with the other constructions regarding the external world which the mind had already made and to which it had committed itself. The first alternative would not for that reason have been 'false'. For the beliefs regarding the external world with which it conflicts were themselves not 'facts' but simply constructions, which the mind was under no compulsion to adopt, and which it could drop, if it wished, in order to embrace the first alternative regarding the motion of the billiard ball. But the first alternative could not be incorporated into the body of knowledge already developed. If the mind had adopted it, it would have had to undo practically the whole of its previous work. It would have involved a reconstruction of practically the whole of our knowledge of the external world on lines totally different from those along which that knowledge had actually developed. Therefore, although the two explanations must both be regarded as 'true' in themselves, yet the mind has had, in the interests of consistency, definitely to accept one and reject the other.

The three geometrical axioms are in the same way alternative truths. But they differ from the example considered in the last paragraph in one interesting and important respect. In that case the mind had definitely to choose one alternative and reject the other. But it is not necessary for the mind to accept one of the geometries and reject the other two. Mathematicians are not divided into three hostile armies flying the banners of Euclid, Riemann, and Lobachevsky. They accept all the geometries and use on each occasion the one which suits that occasion. We could only incorporate into human knowledge one out of the two alternative explanations of motion. But all three geometries are incorporated into knowledge. What is the reason for this difference?

The reason appears to be quite simple. In the case of the two hypotheses regarding motion, whichever we adopt has logical consequences of the most far-reaching character. If we adopt the hypothesis that motion is mere change of state, this would necessitate radical alterations in the whole field of human knowledge. But the adoption of one or other of the geometrical axioms only involves minor adjustments within the restricted field of geometry itself. It makes no difference to anything outside geometry. Whichever geometry we adopt we shall not have to alter our beliefs that there is a common external world existing independently of perception, that tactile and visual space are identical, that 'things' exist and have qualities. Much less shall we have to alter our botanical, zoological, or historical knowledge.

We may represent the position diagrammatically as follows:

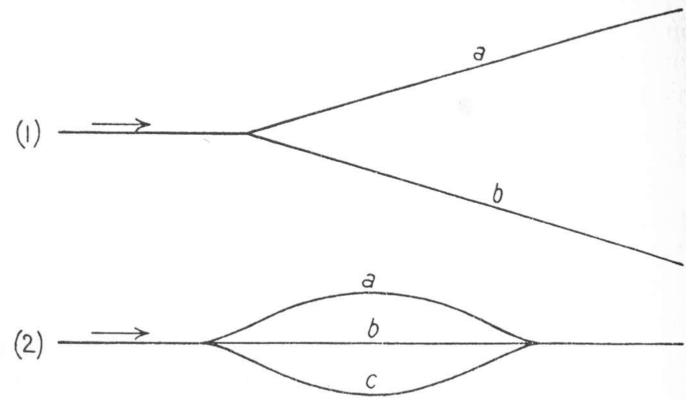

Diagram (1) represents the conditions of choice between the two hypotheses regarding motion. The mind in its evolution may be supposed to move in the direction of the arrow. It reaches at a certain point a parting of the ways. It may either take the road *a* or the road *b*. But the two paths never meet again. Therefore it cannot take both paths. It has to choose one and reject the other.

Diagram (2) shows the conditions of choice between the three geometries which are represented by *a*, *b*, and *c*. They diverge, but meet again. The mind can take whichever path it pleases, and yet afterwards proceed exactly as it would have done if it had taken one of the other paths. You can explain the facts of astronomy, say, either by Euclidean or by non-Euclidean geometry. Whichever course you take you come back to the same point. The rest of science and knowledge remains unaltered. The divergence is only within the restricted field of geometry which is represented by the bulge in diagram (2). The rest of human knowledge is represented by the line before and after the bulge.

But although the mind may select either of the three alternative geometrical truths, it cannot select more than one at a time. It cannot mix up the three, or two of the three, together. Having chosen the path *a* it must proceed along that path until it arrives at the point at which the three paths meet again. It cannot skip across from *a* to *b* or *c* in the middle of the passage. In other words if you begin with the Euclidean axiom you must continue along Euclidean lines. You cannot assume in one and the same astronomical or geometrical problem both that the Euclidean axiom of parallels is true and that the three angles of a triangle are in sum greater than two right angles. Such a procedure would involve a self-contradiction. Hence what compels the mind to take in each case either one or the other of the alternatives, but not more than one, is the laws of logic.

In order to meet the above, the traditional presentation of the laws of logic, and in particular the law of contradiction, may have to be altered. For we see now that three mutually incompatible propositions may all be true. They may be alternative truths. And the law of contradiction as traditionally worded would hardly allow this. Nevertheless the law that we must be self-consistent is preserved in the condition that, if we accept one of the three alternatives, we must accept all the theorems which flow from it, and that we cannot mix up the three systems inconsistently. It does not therefore appear that the law of contradiction is false or that it will have to be radically altered. The spirit of it, namely the necessity that the various propositions which we hold *together* must be mutually compatible and that our thinking must be internally self-consistent, is retained. Probably no more than a rewording of it to suit modern discoveries regarding alternative truths is necessary. To pursue this topic further and to attempt such a restatement belongs to the science of logic, and not to epistemology. And I shall therefore not do so here.

We may now summarize those of our conclusions which are important for the general problems of epistemology.

(1) Mathematical propositions are necessary. But this necessity is due to their being analytical propositions. In other words their necessity is derived from logic.

(2) Hence, our problem being to ascertain whether there is anywhere any such thing as necessary truth which is not derivative, but possesses its necessity in itself, or in other words whether knowledge is tied at its conceptual end as it is at its origin in the given, we must answer that our search of mathematics has not disclosed any such necessary truth or any such tie. For the necessity of mathematics is not in itself. It is merely a case of the necessity of logical laws. There is no such thing as geometrical or mathematical necessity. There may be logical necessity, which applies to mathematics as to everything else. To understand this logical necessity—whether it is real or illusory—and what it implies, we must await our study of logical knowledge in a later chapter.

(3) Mathematical propositions, like other propositions, are either true or false. Mathematical truth is not the same as internal self-consistency, as has sometimes been supposed. It refers to reality outside mathematics.

(4) This outside reality consists in sense-data and concrete things generally. Mathematics is exactly like all other knowledge in this respect. The view that it exists in a world of its own, cut off from concrete things, is false. Geometrical and other mathematical propositions are true when what they assert about concrete realities is true. Mathematics, like all other kinds of knowledge, is tied by the given.

(5) The three geometries of Euclid, Lobachevsky, and Riemann are all true, and equally true. They are alternative truths. Poincaré was mistaken in supposing that to attribute truth to geometry is meaningless.

(6) The three axioms which lie at the bases of the three geometries are existential constructions of the ordinary type, and are similar in all ways to the existential constructions which are involved in our everyday knowledge of things in the external world.